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On Rigorous Mathematical Definitions of Correlation Dimension and Generalized Spectrum for Dimensions

Ya. B. Pesin[1]

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We consider different definitions of the correlation dimension and find some relationships between them and other characteristics of dimension type such as Hausdorff dimension, box dimension, etc. We also introduce different ways to define and study the generalized spectrum for dimensions—a one-parameter family of characteristics of dimension type.

KEY WORDS: Dimension; correlation dimension; Hausdorff dimension; box dimension; generalized spectrum of dimensions.

1. INTRODUCTION

Procaccia and Hentschel[1] described a numerical procedure to introduce a characteristic known now as the correlation dimension. It has become one of the most popular characteristics of dimension type because the algorithm for its calculation is relatively simple and fast. Let us first recall this procedure. Given a time series of points x_1, x_2, x_3,... in a "phase space," define

$$C_k(r) = \frac{2}{k(k-1)} \operatorname{card}\{(i, j)\colon \rho(x_i, x_j) \leqslant r \text{ for } i < j < k\}$$

where card(A) is the number of elements in the set A and ρ denotes the distance between points. Let us now take the limit (assuming that it exists)

$$C(r) = \lim_{k \to \infty} C_k(r)$$

[1] Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802.

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The correlation dimension β is defined to satisfy the asymptotic relation

$$C(r) \sim r^{\beta}$$

for all small enough r (at this point we do not discuss whether this relation can hold at all).

The correlation dimension plays a great role in the numerical investigation of different models, including strange attractors.[13,16] Consider a dynamical system (f, X) where X is a separable metric space with a distance ρ, and $f: X \to X$ is a continuous map. From now on we assume that X has a finite topological dimension. Usually X is to be a Euclidean space or a smooth Riemannian manifold. Given $x \in X$, we can apply the above procedure to the sequence of points $x_n = f^n(x)$. If x is "typical" with respect to a measure μ invariant under f (usually, μ is supposed to be ergodic), then β represents the correlation dimension of the system (X, f, μ).

The above procedure can be treated from a rigorous mathematical viewpoint in different ways. In this paper we consider three of them. The first one directly follows from the above "physical" description. The second is due to D. Ruelle (unpublished). We will show that under certain conditions they produce the same result.

The correlation dimension defined in these ways in general does not coincide with other well-known characteristics of dimension type such as the Hausdorff dimension, box dimension (known also as capacity), information dimension, etc. We introduce another definition of the correlation dimension which is a modification of the previous ones but is not equivalent to them. However, it can still serve in the above numerical procedure and has advantage of being equivalent under certain natural conditions to other characteristics of dimension type.

We also consider different ways to introduce a one-parameter family of characteristics of dimension type called the generalized spectrum for dimensions. The first one was suggested by Hentschel and Proccacia[12] as a physical scale of dimensions generalizing the correlation dimension. We introduce a modification of this approach which seems to be more productive. It is also equivalent to another approach presented in ref. 10 and based on a general construction due to Carathéodory. We show that under certain conditions the spectrum does not essentially depend on the parameter.

2. SOME BASIC DEFINITIONS AND RESULTS

2.1. Hausdorff Dimension

Consider a set $Z \subset X$ and define its α-Hausdorff measure ($\alpha \geqslant 0$ is a real number) as[2,3]

$$m_H(Z, \alpha) = \lim_{\delta \to 0} \inf_G \left\{ \sum_{U \in G} (\text{diam } U)^\alpha \colon \bigcup_{U \in G} U \supset Z, U \leqslant \varepsilon \right\}$$

where G is a finite or countable collection of open sets of diameter $\leqslant \varepsilon$ covering Z (it is easy to see that the above limit exists). The function $m_H(Z, \cdot)$ has the following property: there exists an overchanged value α_H such that $m_H(Z, \alpha) = \infty$ for $\alpha < \alpha_H$ and $m_H(Z, \alpha) = 0$ for $\alpha > \alpha_H$. The quantity

$$\begin{aligned} \dim_H Z = \alpha_H &= \inf\{\alpha \colon m_H(Z, \alpha) = 0\} \\ &= \sup\{\alpha \colon m_H(Z, \alpha) = \infty\} \end{aligned}$$

is called the Hausdorff dimension of the set Z.

2.2. Box Dimension

Let us define upper and lower α-box measures of a set $Z \subset X$ by setting, respectively,

$$\bar{m}(Z, \alpha) = \overline{\lim_{\varepsilon \to 0}} \inf_G \left\{ \sum_{U \in G} (\text{diam } U)^\alpha \colon \bigcup_{U \in G} U \supset Z, \text{diam } U = \varepsilon \right\}$$

$$\underline{m}(Z, \alpha) = \underline{\lim}_{\varepsilon \to 0} \inf_G \left\{ \sum_{U \in G} (\text{diam } U)^\alpha \colon \bigcup_{U \in G} U \supset Z, \text{diam } U = \varepsilon \right\}$$

where G is a finite or countable covering of Z by balls of radius ε. The functions $\bar{m}(Z, \cdot)$, $\underline{m}(Z, \cdot)$ have the following property: there exist $\bar{\alpha}$, $\underline{\alpha}$ such that $\bar{m}(Z, \alpha) = \infty$ for $\alpha < \bar{\alpha}$ [$\underline{m}(Z, \alpha) = \infty$ for $\alpha < \underline{\alpha}$] and $\bar{m}(Z, \alpha) = 0$ for $\alpha > \bar{\alpha}$ [$\underline{m}(Z, \alpha) = 0$ for $\alpha > \underline{\alpha}$]. The quantities

$$\bar{C}(Z) = \bar{\alpha}, \qquad \underline{C}(Z) = \underline{\alpha}$$

are called the upper and lower box dimensions of Z, respectively (the other names often used are lower and upper capacities). One can prove that

$$\bar{C}(Z) = \overline{\lim_{\varepsilon \to 0}} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)}, \qquad \underline{C}(Z) = \underline{\lim}_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{\log(1/\varepsilon)}$$

where $N(Z, \varepsilon)$ is the least number of balls of radius ε that cover the set Z. It follows directly from the definitions that

$$\dim_H Z \leqslant \underline{C}(Z) \leqslant \bar{C}(Z)$$

2.3. Dimension Characteristics of a Measure

Let μ be a Borel probability measure on X. The quantities

$$\dim_H \mu = \inf\{\dim_H Z: Z \subset X, \mu(Z) = 1\}$$

$$\bar{C}(\mu) = \lim_{\delta \to 0} \inf\{\bar{C}(Z): Z \subset X, \mu(Z) \geqslant 1 - \delta\}$$

$$\underline{C}(\mu) = \lim_{\delta \to 0} \inf\{\underline{C}(Z): Z \subset X, \mu(Z) \geqslant 1 - \delta\}$$

are called, respectively, the Hausdorff measure dimension and the upper and lower box measure dimensions.[4]

Given $\varepsilon > 0$ and $\delta > 0$, we denote by $N(\varepsilon, \delta)$ the least number of balls of radius ε that are necessary to cover a set of μ-measure $\geqslant (1 - \delta)$. The quantities

$$\bar{C}_L(\mu) = \lim_{\delta \to 0} \overline{\lim_{\varepsilon \to 0}} \frac{\log N(\varepsilon, \delta)}{\log(1/\varepsilon)}$$

$$\underline{C}_L(\mu) = \lim_{\delta \to 0} \underline{\lim}_{\varepsilon \to 0} \frac{\log N(\varepsilon, \delta)}{\log(1/\varepsilon)}$$

are called, respectively, the upper and lower Ledrappier measure dimensions.[4]

Let ξ be a finite partition of X. We set

$$H_\mu(\xi) = -\sum \mu(C_\xi) \log \mu(C_\xi)$$

where the sum is taken over all elements C_ξ of the partition ξ. Then we define

$$H_\mu(\varepsilon) = \inf_\xi \{H_\mu(\xi): \operatorname{diam} \xi \leqslant \varepsilon\}$$

(here $\operatorname{diam} \xi = \max_{C_\xi} \operatorname{diam} C_\xi$). Finally, we introduce quantities

$$\bar{R}(\mu) = \overline{\lim_{\varepsilon \to 0}} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)}, \qquad \underline{R}(\mu) = \underline{\lim}_{\varepsilon \to 0} \frac{H_\mu(\varepsilon)}{\log(1/\varepsilon)}$$

called, respectively, the upper and lower information measure dimensions (or upper and lower Renyi measure dimensions).[4]

Given $x \in X$, we set

$$\bar{d}_\mu(x) = \overline{\lim_{\delta \to 0}} \frac{\log \mu(B(x, \delta))}{\log \delta}$$

$$\underline{d}_\mu(x) = \underline{\lim}_{\delta \to 0} \frac{\log \mu(B(x, \delta))}{\log \delta}$$

[here $B(x, \delta)$ is the ball of radius δ centered at x]. They are called the upper and lower pointwise measure dimensions at x (or the upper and lower local dimensions at x).

We formulate now the basic results establishing the connections between the quantities introduced above.

Proposition 1.[4] 1. $\dim_H \mu \leqslant \underline{C}_L(\mu) \leqslant \underline{C}(\mu) \leqslant \bar{C}(\mu)$.

2. $\underline{C}_L(\mu) \leqslant \bar{C}_L(\mu) \leqslant \bar{C}(\mu)$.

3. $\underline{R}(\mu) \leqslant \bar{R}(\mu)$; $\underline{d}_\mu(x) \leqslant \bar{d}_\mu(x)$.

Proposition 2.[4] Assume that for μ-almost every $x \in X$

$$\underline{d} \leqslant \underline{d}_\mu(x) \leqslant \bar{d}_\mu(x) \leqslant \bar{d} \tag{1}$$

where, $\underline{d}$, $\bar{d}$ are two constants independent of x. Then

$$\underline{d} \leqslant \dim_H \mu \leqslant \underline{C}(\mu) \leqslant \bar{C}(\mu) \leqslant \bar{d}$$

This statement implies the following result.

Proposition 3.[4] Assume that for μ-almost every $x \in X$

$$\underline{d}_\mu(x) = \bar{d}_\mu(x) = d \tag{2}$$

(d does not depend on x). Then

$$\dim_H \mu = \underline{C}_L(\mu) = \bar{C}_L(\mu) = \underline{C}(\mu) = \bar{C}(\mu) = \underline{R}(\mu) = \bar{R}(\mu) = d$$

2.4. The relation (2) does not necessarily hold. Ledrappier and Misiurewicz[6] constructed an example of a smooth map on $[0, 1]$ preserving an ergodic measure μ for which (2) is violated on a set of points of a positive measure. An example of another nature was demonstrated by Cutler.[7] It is a one-dimensional continuous map for which $\underline{d}_\mu(x) = \bar{d}_\mu(x) = d(x)$ for μ-almost all x (μ is an invariant ergodic measure), but the common value $d(x)$ essentially depends on x.

On the other hand, there is a conjecture due to Eckmann and Ruelle[8] claiming that if f is a diffeomorphism of class C^2 on a compact Riemannian manifold X preserving an ergodic Borel measure μ with nonzero Lyapunov exponents (for definition see, for example, ref. 9), then (2) holds for μ-almost every $x \in X$. L. S. Young proved that this is true in the two-dimensional case.

Proposition 4.[4] Let X be a two-dimensional, compact, smooth Riemannian manifold, $f: X \to X$ a C^2-diffeomorphism preserving a Borel ergodic measure μ with nonzero Lyapunov exponents $\chi_\mu^{(1)} > 0 > \chi_\mu^{(2)}$. Then, for μ-almost every $x \in X$,

$$\underline{d}_\mu(x) = \bar{d}_\mu(x) = d = h_\mu(f)\left(\frac{1}{\chi_\mu^{(1)}} - \frac{1}{\chi_\mu^{(2)}}\right)$$

where $h_\mu(f)$ is the Kolmogorov–Sinai entropy of f.

3. THE FIRST DEFINITION OF THE CORRELATION DIMENSION

3.1. Let X be a metric space with a distance ρ, $f: X \to X$ a continuous map preserving a Borel normalized measure μ. Given $x \in X$, define

$$C(x, n, r) = \frac{2}{n^2} \operatorname{card}\{(i, j): \rho(f^i(x), f^j(x)) \leqslant r \text{ for } 0 \leqslant i \leqslant j < n\}$$

Definition 1. The quantities

$$\bar{\beta}(x) = \varlimsup_{r \to 0} \lim_{n \to \infty} \frac{\log C(x, n, r)}{\log r}$$

$$\underline{\beta}(x) = \varliminf_{r \to 0} \lim_{n \to \infty} \frac{\log C(x, n, r)}{\log r}$$

are called, respectively, the upper and lower correlation dimensions at point x (so far we have assumed that the limit when $n \to \infty$ exists).

It is easy to see that $\underline{\beta}(x) \leqslant \bar{\beta}(x)$. If for μ-almost every $x \in X$

$$\underline{\beta}(x) = \bar{\beta}(x) \overset{\text{def}}{=} \beta$$

(β does not depend on x), then it is called the correlation dimension of the dynamical system (X, f, μ). Now we consider the existence of the limit when $n \to \infty$.

Theorem 1. Assume that μ is an ergodic measure. There exists a countable set $Q \subset \mathbb{R}^1$ such that for any $r \notin Q$ and μ-almost every $x \in X$,

$$\lim_{n \to \infty} C(x, n, r) = \int_X \mu(B(y, r))\, d\mu(y) \overset{\text{def}}{=} \varphi(r)$$

Proof. For $x \in X$, $A \subset X$, $n > 0$, and $m \geqslant 0$ denote by $N(x, A, n, m)$ the number of points $f^i(x)$, $-m \leqslant i \leqslant n$, for which $f^i(x) \in A$. We will also use the shorter notation $N(x, A, n) \equiv N(x, A, n, 0)$. Let us fix a countable number of closed balls $B_k = B(y_k, r_k)$, $k \geqslant 0$, forming a basis of topology in X. Since μ is ergodic, there exists a set $Y \subset X$ of full measure such that for any $x \in Y$, $k \geqslant 0$,

$$\lim_{n \to \infty} \frac{N(x, B_k, n, m)}{n+m} = \mu(B_k)$$

It is easy to see that the function $\varphi(r)$ is monotonic and, hence, can have not more than a countable set of discontinuity points. We denote this set by D_φ and let $C_\varphi = \mathbb{R}^1 \backslash D_\varphi$. One can verify that for any $r \in C_r$ and any $x \in X$

$$\mu(\partial B(x, r)) = 0$$

Fix $x \in Y$, $r \in C_\varphi$ and choose sequences $y_k^{(1)}$, $r_k^{(1)}$ and $y_k^{(2)}$, $r_k^{(2)}$ such that $y_k^{(1)} \to x$, $y_k^{(2)} \to x$, $r_k^{(1)} \to r$, $r_k^{(2)} \to r$, and

$$B_k^{(1)} \equiv B(y_k^{(1)}, r_k^{(1)}) \subset B(x, r) \subset B(y_k^{(2)}, r_k^{(2)}) \equiv B_k^{(2)}$$

We also can assume that $\mu(\partial B_k^{(i)}) = 0$, $i = 1, 2$, $k \geqslant 0$. This implies that

$$\mu(B_k^{(1)}) \leqslant \mu(B(x, r)) \leqslant \mu(B_k^{(2)})$$

and

$$\mu(B_k^{(i)}) \to \mu(B(x, r)), \qquad i = 1, 2$$

In other words, given $\varepsilon > 0$, one can find $k(\varepsilon) > 0$ such that for any $k \geqslant k(\varepsilon)$, $i = 1, 2$,

$$|\mu(B_k^{(i)}) - \mu(B(x, r))| \leqslant \varepsilon$$

Let us now fix $k \geqslant k(\varepsilon)$. There exists $n_\varepsilon^{(1)}$ such that for any $n \geqslant 0$, $m \geqslant 0$, and $n + m \geqslant n_\varepsilon^{(1)}$,

$$\mu(B_k^{(i)}) - \varepsilon \leqslant \frac{N(x, B_k^{(i)}, n, m)}{n+m} \leqslant \mu(B_k^{(i)}) + \varepsilon, \qquad i = 1, 2$$

This produces, for $m + n \geqslant n_\varepsilon^{(1)}$,

$$\frac{N(x, B(x, r), n, m)}{n+m} \leqslant \frac{N(x, B_k^{(2)}, n, m)}{n+m} \leqslant \mu(B_k^{(2)}) + \varepsilon \leqslant \mu(B(x, r)) + 2\varepsilon$$

and

$$\frac{N(x, B(x, r), n, m)}{n+m} \geqslant \frac{N(x, B_k^{(1)}, n, m)}{n} \geqslant \mu(B_k^{(1)}) - \varepsilon \geqslant \mu(B(x, r)) - 2\varepsilon$$

This means that for any $x \in Y$, $r \in C_\varphi$, $n \geqslant 0$, $m \geqslant 0$, and $n+m \geqslant n_\varepsilon^{(1)}$,

$$\left| \frac{N(x, B(x, r), n, m)}{n+m} - \mu(B(x, r)) \right| \leqslant 2\varepsilon$$

Using again the fact that μ is ergodic, we have for μ-almost every $x \in Y$,

$$\lim_{n \to \infty} \frac{N(x, Y, n)}{n} = \mu(Y)$$

Denote by $\tilde{Y}$ the set of such points. We have that $\mu(\tilde{Y}) = 1$.

Let us fix $r \in C_\varphi$. Since μ is ergodic and $\mu(B(x, r))$ is a bounded Borel function on X, we have by virtue of the Birkhoff ergodic theorem that for μ-almost every $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \mu(B(f^i(x), r)) = \int_X \mu(B(y, r))\, d\mu(y)$$

This implies the existence of a measurable set $X_\varepsilon \subset X$ with $\mu(X_\varepsilon) \geqslant 1 - \varepsilon$ and a number $n_\varepsilon^{(2)} > 0$ such that for any $x \in X_\varepsilon$, $n \geqslant n_\varepsilon^{(2)}$,

$$\left| \frac{1}{n} \sum_{i=1}^{n-1} \mu(B(f^i(x), r)) - \int_X \mu(B(y, r))\, d\mu(y) \right| \leqslant \varepsilon$$

Moreover, for μ-almost every $x \in X_\varepsilon$

$$\lim_{n \to \infty} \frac{N(x, X_\varepsilon, n)}{n} = \mu(X_\varepsilon)$$

Denote by $\tilde{X}_\varepsilon$ the set of such points. One can see that $\mu(\tilde{X}_\varepsilon) = \mu(X_\varepsilon) \geqslant 1 - \varepsilon$. Set now $\tilde{X}_\varepsilon^{(1)} = \tilde{X}_\varepsilon \cap \tilde{Y}$, $n_\varepsilon = \max\{n_\varepsilon^{(1)}, n_\varepsilon^{(2)}\}$. We have $\mu(\tilde{X}_\varepsilon^{(1)}) \geqslant 1 - \varepsilon$.

Let us fix $x \in \tilde{X}_\varepsilon^{(1)}$, $n \geqslant n_\varepsilon$. Write $n = l n_\varepsilon + n_0$ with $0 \leqslant n_0 < n_\varepsilon$. Denote also

$$N_i = N(f^i(x), B(f^i(x), r), n-i, i), \qquad i \geqslant 0$$

One can rewrite the expression for $C(x, n, r)$ in the form

$$\begin{aligned} C(x, n, r) &= \frac{1}{n^2} \sum_{i=0}^{n} \frac{N_i}{n} \\ &= \frac{1}{n^2} \sum{}' \frac{N_i}{n} + \frac{1}{n^2} \sum{}'' \frac{N_i}{n} + \frac{1}{n^2} \sum{}''' \frac{N_i}{n} \end{aligned}$$

where the first sum is taken over all i, $1 \leqslant i \leqslant ln_\varepsilon$, with $f^i(x) \in \tilde{X}_\varepsilon^{(1)}$, the second one over all i, $ln_\varepsilon + 1 \leqslant i \leqslant n$, and the third one over all other i. In order to estimate the second sum, let us notice that $N_i \leqslant n$. This implies that

$$\frac{1}{n} \sum'' \frac{N_i}{n} \leqslant \frac{n_\varepsilon}{n} \leqslant \varepsilon$$

if n is big enough. Let us estimate the third sum. Taking into account that $N_i/n \leqslant 1$, we have that

$$\frac{1}{n} \sum''' \frac{N_i}{n} \leqslant \frac{1}{n} [n - N(x, \tilde{X}_\varepsilon^{(1)}, n)] \leqslant 2\varepsilon$$

if n is large enough.

Now consider the first sum. It follows from what was said above that for all sufficiently big n

$$\left| \frac{1}{n} \sum' \frac{N_i}{n} - \int_X \mu(B(y, r))\, d\mu(y) \right|$$
$$\leqslant \left| \frac{1}{n} \sum' \frac{N_i}{n} - \frac{1}{n} \sum_{i=1}^{n-2} \mu(B(f^i(x), r)) \right| + \varepsilon$$
$$\leqslant \frac{1}{n} \sum' \left| \frac{N_i}{n} - \mu(B(f^i(x), r)) \right| + 2\varepsilon \leqslant 3\varepsilon$$

The desired result follows now from the Borel–Cantelli lemma. ∎

Remark. The arguments in the proof of Theorem 1 show that the exceptional set $Q = D_\varphi$.

3.2. Now we can obtain formulas for upper and lower correlation dimensions. We shall do this under an additional assumption. The general case is considered in ref. 17.

Theorem 2. Assume that μ is an ergodic Borel measure satisfying the following property:

(A) The function $\varphi(r)$ is continuous on an interval $[0, r_0]$, $r_0 > 0$.

Then for μ-almost every $x \in X$,

$$\bar{\beta}(x) = \varlimsup_{r \to 0} \frac{\log \int_X \mu(B(y, r))\, d\mu(y)}{\log r}$$

$$\underline{\beta}(x) = \varliminf_{r \to 0} \frac{\log \int_X \mu(B(y, r))\, d\mu(y)}{\log r}$$

Proof. Let us fix $\varepsilon > 0$. Assumption (A) implies that there exists $\delta > 0$ such that $|\varphi(r_1) - \varphi(r_2)| \leqslant \varepsilon$ for any r_1, $r_2 \in [0, r_0)$ with $|r_1 - r_2| < \delta$. It follows from Theorem 1 that given $r > 0$, there is a set $Y_r \subset X$ of full measure such that for any $x \in Y_r$ the limit exists

$$\lim_{n \to \infty} C(x, n, r) = \varphi(r)$$

Let us fix a countable, everywhere dense set $T \subset \mathbb{R}^1$ and put $Y = \bigcap_{r \in T} Y_r$. It is easy to see that $\mu(Y) = 1$. Given $r \in \mathbb{R}^1$, one can find r_1, $r_2 \in T$ satisfying $r_1 < r < r_2$, $r_2 - r_1 \leqslant \delta$. It is easy to see that for any $n > 0$,

$$C(x, n, r_1) \leqslant C(x, n, r) \leqslant C(x, n, r_2)$$

If n is big enough, we also have that

$$C(x, n, r_1) \geqslant \varphi(r_1) - \varepsilon \geqslant \varphi(r) - 2\varepsilon$$

$$C(x, n, r_2) \leqslant \varphi(r_2) + \varepsilon \leqslant \varphi(r) - 2\varepsilon$$

The above inequalities imply that $|C(x, n, r) - \varphi(r)| \leqslant 2\varepsilon$ if n is sufficiently large, which leads to the desired result. ∎

4. RUELLE'S APPROACH

4.1. We give another definition of the correlation dimension using an approach suggested by D. Ruelle.

Let X be a metric space with a distance ρ and μ a Borel normalized measure on X. Consider $Y = X \times X$ with the metric $\bar{\rho}$ $[\bar{\rho}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2)]$, the measure $\nu = \mu \times \mu$, and let Δ be the diagonal, $\Delta = \{y \in Y\colon y = (x, x), x \in X\}$. Given $r > 0$, denote by $U(\Delta, r) = \{y \in Y\colon \bar{\rho}(y, \Delta) \leqslant r\}$.

Definition 2. The quantities

$$\bar{\gamma} = \bar{\gamma}(\mu) = \overline{\lim_{r \to 0}} \frac{\log \nu(U(\Delta, r))}{\log r}$$

$$\underline{\gamma} = \underline{\gamma}(\mu) = \underline{\lim}_{r \to 0} \frac{\log \nu(U(\Delta, r))}{\log r}$$

are called, respectively, the upper and lower correlation dimensions of measure μ.

It is easy to see that $\underline{\gamma}(\mu) \leqslant \bar{\gamma}(\mu)$. If they are equal to each other, the common value $\gamma = \underline{\gamma}(\mu) = \bar{\gamma}(\mu)$ is called the correlation dimension.

It is not difficult to verify, using the definition of the metric $\bar{\rho}$, that

$$U(\Delta, r) = \bigcup_{X} (x, B(x, r))$$

We have also that $(x_1, B(x_1, r)) \cap (x_2, B(x_2, r)) = \varnothing$ if $x_1 \neq x_2$. Since ν is the direct product measure, the above arguments imply that

$$\nu(U(A, r)) = \int_X \mu(B(x, r))\, d\mu(x)$$

Now one can rewrite the definition of the correlation dimension in the following form:

$$\bar{\gamma}(\mu) = \overline{\lim_{r \to 0}} \frac{1}{\log r} \log \int_X \mu(B(x, r))\, d\mu(x)$$

$$\underline{\gamma}(\mu) = \underline{\lim}_{r \to 0} \frac{1}{\log r} \log \int_X \mu(B(x, r))\, d\mu(x)$$

4.2. The definitions of the upper and lower correlation dimensions $\bar{\gamma}$, $\underline{\gamma}$ do not involve any dynamics. However, they also can be interpreted from a "dynamical" point of view. Namely, let $f: X \to X$ be a continuous map preserving a Borel normalized ergodic measure μ. Given $x, y \in X$, define

$$C(x, y, n, r) = \frac{2}{n(n-1)} \operatorname{card}\{(i, j): \rho(f^i(x), f^j(y)) \leqslant r \text{ for } 0 \leqslant i < j \leqslant n\}$$

Consider the space $Y = X \times X$ with the metric $\bar{\rho}$ $[\bar{\rho}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2)]$, the measure $\nu = \mu \times \mu$, and the $\mathbb{Z}^2$ action given by

$$f_{(i,j)}(x, y) = (f^i(x), f^j(y))$$

One can verify that ν is invariant and ergodic for this action. This implies that for ν-almost every pair (x, y) there exists the limit

$$\lim_{n \to \infty} C(x, y, n, r) = \nu(U(A, r))$$

This justifies that for ν-almost every pair (x, y),

$$\underline{\gamma} = \underline{\lim}_{r \to 0} \lim_{n \to \infty} \frac{\log C(x, y, n, r)}{\log r}$$

$$\bar{\gamma} = \overline{\lim_{r \to 0}} \lim_{n \to \infty} \frac{\log C(x, y, n, r)}{\log r}$$

Assume in addition that μ satisfies condition (A). Theorem 2 implies that for μ-almost every $x \in X$,

$$\bar{\gamma}(\mu) = \bar{\beta}(x), \qquad \underline{\gamma}(\mu) = \underline{\beta}(x)$$

This gives another "dynamical" interpretation of $\bar{\gamma}$, $\underline{\gamma}$ and also shows that the two definitions of the correlation dimension coincide.

5. LIMIT CORRELATION DIMENSION

The correlation dimension defined above recently has been the subject of intensive study (see, for example, refs. 13, 15, and 16). Although it is convenient to use it for the numerical analysis of systems, more careful investigation shows that it cannot be completely considered a characteristic of dimension type. We suggest here a modification of the definition of the correlation dimension. It can still serve in the numerical procedure described above, but it is more dimensionlike.

Definition 3. The quantities

$$\bar{\alpha} = \bar{\alpha}(\mu) = \lim_{\delta \to 0} \sup \overline{\lim_{r \to 0}} \log \int_Z \mu(B(x, r))\, d\mu(x)/\log r$$

$$\underline{\alpha} = \underline{\alpha}(\mu) = \lim_{\delta \to 0} \sup \underline{\lim}_{r \to 0} \log \int_Z \mu(B(x, r))\, d\mu(x)/\log r$$

are called, respectively, the upper and lower limit correlation dimensions of measure μ [here sup is taken over all sets $Z \subset X$ with $\mu(Z) \geqslant 1 - \delta$; it is also easy to see that the limit exists when $\delta \to 0$].

Obviously, $\underline{\alpha}(\mu) \leqslant \bar{\alpha}(\mu)$. If they are equal to each other, the common value is called the limit correlation dimension. It follows directly from the definitions that

$$\underline{\alpha} \geqslant \underline{\gamma}, \qquad \bar{\alpha} \geqslant \bar{\gamma}$$

In general, one cannot expect better relationships between these characteristics even under certain "natural" assumptions (see below and ref. 14). However, the limit correlation dimension has close relationships to the other characteristics of dimension type. We formulate this in the following statements; the proof will be given later (Theorems 4–6).

Theorem 3. If μ satisfies condition (2), then $\alpha = \underline{\alpha} = \bar{\alpha} = d = \underline{C}(\mu) = \bar{C}(\mu) = \underline{C}_L(\mu) = \bar{C}_L(\mu) = \underline{R}(\mu) = \bar{R}(\mu) = \dim_H \mu$.

6. GENERALIZED SPECTRUM FOR DIMENSIONS

6.1. Following Hentschel and Procaccia,[12] one can generalize the definition of the correlation dimension by introducing a one-parameter family of characteristics called the generalized spectrum for dimensions. Namely, given $q>0$, define

$$\bar{\gamma}_q(\mu)=\frac{1}{q}\varlimsup_{r\to 0}\frac{1}{\log r}\log\int_X \mu(B(x,r))^q\,d\mu(x)$$

$$\underline{\gamma}_q(\mu)=\frac{1}{q}\varliminf_{r\to 0}\frac{1}{\log r}\log\int_X \mu(B(x,r))^q\,d\mu(x)$$

If $\underline{\gamma}_q(\mu)=\bar{\gamma}_q(\mu)$ for any $q>0$, the common value $\gamma_q(\mu)$ is called the generalized spectrum for dimensions. It is easy to see that $\underline{\gamma}_1(\mu)=\underline{\gamma}(\mu)$ and $\bar{\gamma}_1(\mu)=\bar{\gamma}(\mu)$.

We consider a modification of the above definition and also extend it to q such that $-1\leqslant q<0$. Let

$$X_0=\{x\in X\colon \mu(B(x,r))>0 \text{ for any } r>0\}$$

One can see that $\mu(X_0)=1$. Given $\delta>0$, take $Z\subset X_0$ such that $\mu(Z)\geqslant 1-\delta$. For q, $-1\leqslant q<0$, $q>0$, define

$$\bar{\alpha}_q(\mu)=\frac{1}{q}\lim_{\delta\to 0}\sup\varlimsup_{r\to 0}\log\int_Z \mu(B(x,r))^q\,d\mu(x)/\log r$$

$$\underline{\alpha}_q(\mu)=\frac{1}{q}\lim_{\delta\to 0}\sup\varliminf_{r\to 0}\log\int_Z \mu(B(x,r))^q\,d\mu(x)/\log r$$

where the sup is taken over all sets $Z\subset X_0$ with $\mu(Z)\geqslant 1-\delta$ [for q, $-1\leqslant q<0$, we do not exclude the case when $\bar{\alpha}_q(\mu)$ and $\underline{\alpha}_q(\mu)$ are infinite]. We have that $\underline{\alpha}_q(\mu)\leqslant\bar{\alpha}_q(\mu)$ if $q>0$ and $\underline{\alpha}_q(\mu)\geqslant\bar{\alpha}_q(\mu)$ if $-1\leqslant q<0$. If $\underline{\alpha}_q(\mu)=\bar{\alpha}_q(\mu)$ for any $q\geqslant -1$, $q\neq 0$, the common value $\alpha_q(\mu)$ is called the limit generalized spectrum for dimensions. In general, $\underline{\alpha}_q(\mu)\geqslant\underline{\gamma}_q(\mu)$ and $\bar{\alpha}_q(\mu)\geqslant\bar{\gamma}_q(\mu)$ if $q>0$, and $\underline{\alpha}_q(\mu)<\underline{\gamma}_q(\mu)$ and $\bar{\alpha}_q(\mu)\leqslant\bar{\gamma}_q(\mu)$ if $-1\leqslant q<0$, and we do not expect equalities. It is also easy to see that $\underline{\alpha}_1(\mu)=\underline{\alpha}$, $\bar{\alpha}_1(\mu)=\bar{\alpha}$.

6.2. In ref. 10 we suggested another definition of the limit generalized spectrum for dimensions based on a general construction essentially due to Carathéodory. We given here a brief description of this approach.

Let X be a metric space and μ a normalized Borel measure on X. Fix λ, $q \in \mathbb{R}$, $q \geqslant -1$, $q \neq 0$, consider a set $Z \subset X$, and define its (λ, q)-measure as

$$m_q(Z, \lambda) = \lim_{\varepsilon \to 0} \inf_G \left\{ \sum_{U \in G} \mu(U)^{q+1} (\operatorname{diam} U)^{\lambda} : \bigcup_{U \in G} U \subset Z, \operatorname{diam} U \leqslant \varepsilon \right\}$$

where G is a finite or countable collection of open sets of diameter $\leqslant \varepsilon$ covering Z. The function $m_q(Z, \cdot)$ has the following property: there exists an overchanged value $\lambda_q = \lambda_q(Z)$ such that $m_q(Z, \lambda) = 0$ for $\lambda > \lambda_q$ and $m_q(Z, \lambda) = \infty$ for $\lambda < \lambda_q$. The value $-(1/q)\,\lambda_q$ is called the q-dimension of Z and is denoted by $\dim_q Z$. It is easy to see that $\dim_{-1} Z = \dim_H Z$.

We now change the above definition in order to introduce the q-box dimension. Set

$$\bar{m}_q(Z, \lambda) = \overline{\lim_{\varepsilon \to 0}} \inf_G \left\{ \sum_{U \in G} \mu(U)^{q+1} (\operatorname{diam} U)^{\lambda} : \bigcup_{U \in G} U \supset Z, \operatorname{diam} U = \varepsilon \right\}$$

$$\underline{m}_q(Z, \lambda) = \underline{\lim}_{\varepsilon \to 0} \inf_G \left\{ \sum_{U \in G} \mu(U)^{q+1} (\operatorname{diam} U)^{\lambda} : \bigcup_{U \in G} U \supset Z, \operatorname{diam} U = \varepsilon \right\}$$

Here G is a finite or countable covering of Z by balls of radius ε. The functions $\bar{m}_q(Z, \cdot)$ and $\underline{m}_q(Z, \cdot)$ have the following property: there exist $\bar{\lambda}_q = \bar{\lambda}_q(Z)$ and $\underline{\lambda}_q = \underline{\lambda}_q(Z)$ such that $\bar{m}_q(Z, \lambda) = 0$ for $\lambda > \bar{\lambda}_q$ $[\underline{m}_q(Z, \lambda) = 0$ for $\lambda > \underline{\lambda}_q]$ and $\bar{m}_q(Z, \lambda) = \infty$ for $\lambda < \bar{\lambda}_q$ $[\underline{m}_q(Z, \lambda) = \infty$ for $\lambda < \underline{\lambda}_q]$. The values $-(1/q)\,\bar{\lambda}_q$ and $-(1/q)\,\underline{\lambda}_q$ are called, respectively, the upper and lower q-box dimensions of Z and are denoted by $\bar{C}_q(Z)$ and $\underline{C}_q(Z)$. One can prove[10] that

$$\bar{C}_q(Z) = \overline{\lim_{\varepsilon \to 0}} \frac{1}{q} \frac{\log \Delta(Z, \varepsilon)}{\log \varepsilon}, \qquad \underline{C}_q(Z) = \underline{\lim}_{\varepsilon \to 0} \frac{1}{q} \frac{\log \Delta(Z, \varepsilon)}{\log \varepsilon}$$

where

$$\Delta(Z, \varepsilon) = \inf_G \left\{ \sum_{B(x_i, \varepsilon) \in G} \mu(B(x_i, \varepsilon))^{q+1} \right\}$$

and G has the above sense. It is easy to see that $\bar{C}_{-1}(Z) = \bar{C}(Z)$ and $\underline{C}_{-1}(Z) = \underline{C}(Z)$.

Finally, let us introduce

$$\dim_q \mu = -\frac{1}{q}\inf\{\lambda_q(Z): \mu(Z)=1\}$$

the q-dimension of measure

$$\bar{C}_q(\mu) = -\frac{1}{q}\lim_{\delta\to 0}\inf\{\bar{\lambda}_q(Z): \mu(Z)\geqslant 1-\delta\}$$

the upper q-box dimension of measure (11)

$$\underline{C}_q(\mu) = -\frac{1}{q}\lim_{\delta\to 0}\inf\{\underline{\lambda}_q(Z): \mu(Z)\geqslant 1-\delta\}$$

the lower q-box dimension of measure

The limit generalized spectrum for dimensions arises when the above three characteristics coincide for all $q\geqslant -1$, $q\neq 0$ as their common value. The main result in ref. 10 claims the following.

Theorem 4. Assume that μ satisfies condition (2). Then

$$\dim_q \mu = \underline{C}_q(\mu) = \bar{C}_q(\mu) = d, \qquad q\geqslant -1, \qquad q\neq 0$$

6.3. We show that the above two definitions of the generalized spectrum for dimensions are equivalent.

Theorem 5. If μ is a Borel probability measure on a metric space X of a finite topological dimension, then for any $q\geqslant 1$ and $Z\subset X$, $\mu(Z)>0$,

$$\bar{C}_q(Z) = \frac{1}{q}\overline{\lim_{r\to 0}}\log\int_Z \mu(B(x,r))^q\,d\mu(x)/\log r$$

$$\underline{C}_q(Z) = \frac{1}{q}\underline{\lim}_{r\to 0}\log\int_Z \mu(B(x,r))^q\,d\mu(x)/\log r$$

Proof. Given $r>0$, let μ_r be the measure on X such that $d\mu_r(x) = \mu(B(x,r))^{q-1}\,d\mu(x)$. Consider the space $Y = X\times X$ with the metric $\bar{\rho}$ and the measure $\nu = \mu\times\mu_r$. Let us take any measurable set $Z\subset X$ and let G be a covering of Z by balls $B(x_i, r)$, $i=1, 2,\ldots$. The sets $\{B(x_i, 2r)\times B(x_i, 2r)\}$ make up the covering of the set

$$Z_r = \bigcup_{x\in Z}(x, B(x,r))\subset Y$$

It is easy to see that

$$\begin{aligned}\int_Z \mu(B(x,r))^q \, d\mu(x) &= \int_Z \mu(B(x,r))\, \mu(B(x,r))^{q-1}\, d\mu(x)\\ &= \nu(Z_r)\\ &\leqslant \sum_{B(x_i,r)\in G} \nu(B(x_i,2r)\times B(x_i,2r))\\ &\leqslant \sum_{B(x_i,r)\in G} \mu(B(x_i,2r)) \int_{B(x_i,2r)} \mu(B(y,2r))^{q-1}\, d\mu(y)\end{aligned}$$

Obviously $B(y, 2r) \subset B(x_i, 4r)$ for $y \in B(x_i, 2r)$. Therefore

$$\int_Z \mu(B(x,r))^q \, d\mu(x) \leqslant \sum_{B(x_i,r)\in G} \mu(B(x_i,4r))^{q+1}$$

Since the above estimation holds for any covering G of Z by balls $B(x_i, r)$, $x_i \in Z$, this implies that

$$\int_Z \mu(B(x,r))^q \, d\mu(x) \leqslant \Delta(Z, 4r) \tag{3}$$

On the other hand, let $Z \subset X_0$ be a set with $\mu(Z) > 0$. Since X has a finite topological dimension, there exists a covering G of Z by balls $B(x_i, r/2)$, $x_i \in Z$, of a finite multiplicity. We have now that

$$\begin{aligned}\sum_{B(x_i,r/2)\in G} \nu(B(x_i,r)\times B(x_i,r)) &\leqslant C_1 \nu\left(\bigcup_{B(x_i,r/2)\in G} B(x_i,r)\times B(x_i,r)\right)\\ &\leqslant C_2 \nu(Z_{2r})\end{aligned}$$

where $C_1 > 0$, $C_2 > 0$ are constants. This implies that

$$\begin{aligned}\Delta\left(Z,\frac{r}{2}\right) &\leqslant \sum_{B(x_i,r/2)\in G} \mu\left(B\left(x_i,\frac{r}{2}\right)\right)^{q+1}\\ &= \sum_{B(x_i,r/2)\in G} \mu\left(B\left(x_i,\frac{r}{2}\right)\right) \mu\left(B\left(x_i,\frac{r}{2}\right)\right)^q\\ &\leqslant \sum_{B(x_i,r/2)\in G} \mu\left(B\left(x_i,\frac{r}{2}\right)\right) \int_{B(x_i,r/2)} \mu(B(y,r))^q\, d\mu(y)\end{aligned}$$

[we use here the fact that $B(x_i, r/2) \subset B(y, r)$ for any $y \in B(x_i, r/2)$]

$$= \sum_{B(x_i, r/2) \in G} \nu(B(x_i, r) \times B(x_i, r)) \leqslant C_2 \nu(Z_{2r})$$

$$= C_2 \int_Z \mu(B(x, 2r))^q \, d\mu(x) \tag{4}$$

The desired result follows now from (4) and (3). ∎

The immediate consequences of Theorem 5 are as follows.

(1) The upper and lower correlation dimensions satisfy

$$\bar{\gamma}(\mu) = \bar{C}_1(X), \qquad \underline{\gamma}(\mu) = \underline{C}_1(X)$$

(2) For $q \geqslant 1$ the generalized spectrum for dimensions satisfies

$$\bar{\gamma}_q(\mu) = \bar{C}_q(X), \qquad \underline{\gamma}_q(\mu) = \underline{C}_q(X)$$

(3) The upper and lower limit correlation dimensions satisfy

$$\bar{\alpha}(\mu) = \bar{C}_1(\mu), \qquad \underline{\alpha}(\mu) = \underline{C}_1(\mu)$$

(4) For $q \geqslant 1$ the limit generalized spectrum for dimensions satisfies

$$\bar{\alpha}_q(\mu) = \bar{C}_q(\mu), \qquad \underline{\alpha}_q(\mu) = \underline{C}_q(\mu)$$

Now we consider the case $-1 \leqslant q < 1$.

Theorem 6. If μ is a Borel probability measure on a metric space X of a finite topological dimension satisfying (1), then for any $q \geqslant -1$, $q \neq 0$,

$$\bar{C}_q(\mu) = \bar{\alpha}_q(\mu), \qquad \underline{C}_q(\mu) = \underline{\alpha}_q(\mu)$$

Proof. By virtue of (2), one can write for μ-almost every $x \in X$ and any small enough r that

$$C_1(x)\, r^{\bar{d}+\alpha} \leqslant \mu(B(x, r)) \leqslant C_2(x)\, r^{\underline{d}-\alpha}$$

where $\alpha > 0$ is a given small number, and $C_1(x) > 0$ and $C_2(x) > 0$ are two Borel functions on X. Given $\delta > 0$, one can find a set X_δ with $\mu(X_\delta) \geqslant 1 - \delta$ and two constants $c_1(\delta) > 0$ and $c_2(\delta) > 0$ such that for any $x \in X_\delta$,

$$C_1(x) \geqslant c_1(\delta), \qquad C_2(x) \leqslant c_2(\delta)$$

Given $r > 0$, let μ_r be the measure on X_δ such that $d\mu_r(x) = \mu(B(x, r))^{q-1}\, d\mu(x)$ (it is easy to see that this measure is correctly defined).

Let now $\nu = \mu \times \mu_r$ be the measure on $X_\delta \times X \subset X \times X$. We extend it to a measure on $X \times X$ by setting $\tilde{\nu}(A) = \nu(A \cap X_\delta \times X)$. Given $\delta > 0$, take any measurable set $Z \subset X_\delta$ with $\mu(Z) \geqslant 1 - \delta$. Repeating arguments in Theorem 5, one can show that (3) and (4) hold. This implies the desired result. ▌

Theorem 7. Assume that μ satisfies condition (2). Then

$$\begin{aligned} d &= \underline{\alpha}_q(\mu) = \bar{\alpha}_q(\mu) = \alpha_q(\mu) \\ &= \underline{C}_q(\mu) = \bar{C}_q(\mu) = \dim_q \mu \\ &= \dim_H \mu = \underline{C}(\mu) = \bar{C}(\mu) = \underline{C}_L(\mu) = \bar{C}_L(\mu) = \underline{R}(\mu) = \bar{R}(\mu) \end{aligned}$$

Remark. For $q = -1$, Theorem 6 produces another definition of the lower and upper box measure dimensions. Namely, if μ satisfies condition (1), then

$$\begin{aligned} \underline{C}(\mu) &= \underline{C}_{-1}(\mu) = \underline{\alpha}_{-1}(\mu) \\ &= -\lim_{\delta \to 0} \sup \varliminf_{r \to 0} \log \int_Z \frac{d\mu(x)}{\mu(B(x, r))} \Big/ \log r \\ \bar{C}(\mu) &= \bar{C}_{-1}(\mu) = \bar{\alpha}_{-1}(\mu) \\ &= -\lim_{\delta \to 0} \sup \varlimsup_{r \to 0} \log \int_Z \frac{d\mu(x)}{\mu(B(x, r))} \Big/ \log r \end{aligned}$$

where sup is taken over all sets Z with $\mu(Z) \geqslant 1 - \delta$.

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REFERENCES

1. I. Procaccia, P. Grassberger, and V. G. E. Hentschel, On the characterization of chaotic motions, in *Lecture Notes in Physics*, No. 179 (Springer, Berlin, 1983), pp. 212–221.
2. P. Billingsley, *Ergodic Theory and Information* (Wiley, New York, 1965).
3. W. Hurewich and H. Wallman, *Dimension Theory* (1941).
4. L.-S. Young, Dimension, entropy and Lyapunov exponents, *Ergod. Theory Dynam. Syst.* **2**:109–124 (1982).

5. A. Renyi, in *Dimension, Entropy and Information* (Transactions Second Prague Conference on Information Theory, Statistical Decision Functions and Random Processes, 1957), pp. 545–556.
6. F. Ledrappier and M. Misiurewicz, Dimension of invariant measures for maps with exponent zero, *Ergod. Theory Dynam. Syst.* **5**:595–610 (1985).
7. C. D. Cutler, Connecting ergodicity and dimension in dynamical systems, *Ergod. Theory Dynam. Syst.* **10**:451–462 (1990).
8. J. P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.* **57**:617–653 (1985).
9. R. Mane, *Ergodic Theory and Differential Dynamics* (Springer-Verlag, Berlin, 1987).
10. Ya. B. Pesin, Generalized spectrum for the dimension: The approach based on Carathéodory's construction, in *Constantin Carathéodory, an International Tribute*, T. M. Rassias, ed. (1991), Vol. 2, pp. 1108–1119.
11. Ya. B. Pesin, Dimensionlike characteristics for the invariant sets of the dynamical systems, *Russ. Math. Surv.* **43**(4) (1988).
12. G. E. Hentsel and I. Procaccia, *Physica* **8D**:435 (1983).
13. P. Frederickson, T. Kaplan, E. Yorke, and G. Yorke, The Lyapunov dimension of strange attractors, *J. Differential Equation* **49**(2):185–206 (1983).
14. E. Ott, T. Sauer, and G. Yorke, Lyapunov partition functions for the dimensions of chaotic sets, *Phys. Rev. A* **39**(8):4212–4222 (1989).
15. C. Grebogi, E. Ott, and G. Yorke, Roundoff-induced periodicity and the correlation dimension of chaotic attractors, *Phys. Rev. A* **38**(7):3688–3692 (1988).
16. T. Farmer, E. Ott, and G. Yorke, The dimension of chaotic attractors, *Physica* **7D**:153–180 (1983).
17. Ya. Pesin and A. Tempelman, Correlation dimension of group action, preprint, Pennsylvania State University (1992).